Quantum Cosmology for the General Bianchi Type II, VI(Class A) and VII(Class A) vacuum geometries

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Abstract

The canonical quantization of the most general minisuperspace actions –i.e. with all six scale factor as well as the lapse function and the shift vector present– describing the vacuum type II, VI and VII geometries, is considered. The reduction to the corresponding physical degrees of freedom is achieved through the usage of the linear constraints as well as the quantum version of the entire set of classical integrals of motion.

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1 Introduction

Since the conception by Einstein of General Relativity Theory, a great many efforts have been devoted by many scientists to the construction of a consistent quantum theory of gravity. These efforts can de divided into two main approaches:

(a) perturbative, in which one splits the metric into a background (kinematical) part and a dynamical one: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and tries to quantize $h_{\mu\nu}$. The only conclusive results existing, are that the theory thus obtained is highly nonrenormalizable [1].

(b) non perturbative, in which one tries to keep the twofold role of the metric (kinematical and dynamical) intact. A hallmark in this direction is canonical quantization.

In trying to implement this scheme for gravity, one faces the problem of quantizing a constrained system. The main steps one has to follow are:

(i) define the basic operators $\widehat{g}_{\mu\nu}$ and $\widehat{\pi}^{\mu\nu}$ and the canonical commutation relation they satisfy.

(ii) define quantum operators $\widehat{H}_\mu$ whose classical counterparts are the constraint functions H_μ.

(iii) define the quantum states $\Psi[g]$ as the common null eigenvector of $\widehat{H}_\mu$, i.e. these satisfying $\widehat{H}_\mu \Psi[g] = 0$. (As a consequence, one has to check that $\widehat{H}_\mu$, form a closed algebra under the basic CCR.)

(iv) find the states and define the inner product in the space of these states.

It is fair to say that the full program has not yet been carried out, although partial steps have been made [2].

In the absence of a full solution to the problem, people have turned to what is generally known as quantum cosmology. This is an approximation to quantum gravity in which one freezes out all but a finite number of degrees of freedom, and quantizes the rest. In this way one is left with a much more manageable problem that is essentially quantum mechanics with constraints. Over the years, many models have appeared in the literature [3]. In most of them, the minisuperspace is flat and the gravitational field is represented by no more than three degrees of freedom (generically the three scale factors of some anisotropic Bianchi Type model [4]).

In order for the article to be as self consistent as possible, we include in section 2, a short introduction to the theory of constrained systems and in section 3, the Kantowski-Sachs model as an interdisciplinary example. In section 4, we present the quantization of the most general Type II, VI(Class A) & VII(Class A) Vacuum Bianchi Cosmologies.

2   Elements of Constrained Dynamics

2.1   Introduction

In these short notes, we present the elements of the general methods and some techniques of the Constrained Dynamics. It is about a powerful mathematical theory (a method, more or less) –primarily developed by P. A. M. Dirac. The scope of it, is to describe singular (the definition is to be presented at the next section) physical systems, using a generalization of the Hamiltonian or the Lagrangian formalism. This theory, is applicable both for discrete (i.e. finite degrees of freedom) and continua (i.e. infinite degrees of freedom) systems.

For the sake of simplicity, the Hamiltonian point view of a physical system is adopted, and the discussion will be restricted on discrete systems. A basic bibliography, at which the interested reader is strongly suggested to consult, is quoted at the end of these notes. Also, the treatment follows reference [5].

2.2   The Hamiltonian Approach

Suppose a discrete physical system, whose action integral is:

$$\mathcal{A} = \int L dt \tag{2.2.2.1}$$

The dynamical coordinates, are denoted by q^i, with $i \in [1, \ldots, N]$ The Lagrangian is a function of the coordinates and the velocities, i.e. $L = L(q^i, \dot{q}^i)$.

A note is pertinent at this point. If one demands the action integral (2.2.2.1), to be scalar under general coordinate transformations (G.C.T.), then he can be sure that the content of the theory to be deduced, will be relativistically covariant even though the form of the deduced equations will not be manifestly covariant, on account of the appearance of one particular time in a dominant place in the theory (i.e. the time variable t occurring already, as soon as one introduces the generalized velocities, in consequently the Lagrangian, and finally the Lagrange transformation, in order to pass from the Lagrangian, to the Hamiltonian).

Variation of the action integral, gives the Euler-Lagrange equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}, \quad i \in [1, \ldots, N] \tag{2.2.2.2}$$

In order to go over to the Hamiltonian formalism, the momentum variables p_i, are introduced through:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad \forall \quad i \tag{2.2.2.3}$$

In the usual dynamical theories, a very restricting assumption is made; that all momenta are independent functions of the velocities, or –in view of the inverse map theorem for a

function of many variables– that the following (Hessian) determinant:

$$|H_{ij}| = |\frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j}| \tag{2.2.2.4}$$

is not zero in the whole domain of its definition. If this is the case, then the theorem guarantees the validity of the assumption, permits to use the Legendre transformation, and the corresponding physical system is called *Regular*. If this is not the case (i.e. some momenta, are not independent functions od the velocities), then there must exist some (say M) independent relations of the type:

$$\phi_m(q, p) = 0, \quad m \in [1, \ldots, M] \tag{2.2.2.5}$$

which are called *Primary Constraints*. The corresponding physical systems, are characterized as *Singular*.

Variation of the quantity $p_i \dot{q}^i - L$ (the Einstein's summation convention is in use), results in:

$$\delta \left(p_i \dot{q}^i - L \right) = \ldots = \left(\delta p_i \right) \dot{q}^i - \left(\frac{\partial L}{\partial q^i} \right) \delta q^i \tag{2.2.2.6}$$

by virtue of (2.2.2.2). One can see that this variation, involves variations of the q's and the p's. So, the quantity under discussion does not involve variation of the velocities and thus can be expressed in terms of the q's and the p's, only. This is the Hamiltonian. It must be laid stress on the fact that the variations, must respect the restrictions (2.2.2.5), i.e. to preserve them –if they are considered as conditions (see, e.g. C. Carathéodory, ''Calculus of Variations and Partial Differential Equations of the First Order'', AMS Chelsea (1989)).

Obviously, the Hamiltonian is not uniquely determined for, zero quantities can be added to it. This means that the following:

$$H_T = H + u^m \phi_m \tag{2.2.2.7}$$

where u^m's are arbitrary coefficients in the phase space (including the time variable), is a valid Hamiltonian too. Variation of (2.2.2.7) results in:

$$\begin{array}{l} \dot{q}^i = \frac{\partial H}{\partial p_i} + u^m \frac{\partial \phi}{\partial p_i} + \text{term that vanishes as (2.2.2.5)} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} - u^m \frac{\partial \phi}{\partial q_i} - \text{term that vanishes as (2.2.2.5)} \end{array} \tag{2.2.2.8}$$

These are the Hamiltonian equations of motion for the system under consideration. This scheme, reflects the previous observation about variations under which, conditions must be preserved.

In order to proceed, a generalization of the Poisson Brackets is needed to be introduced. This is done as follows:
If f, g, h are quantities on a space, endowed with a linear map $\{\ ,\ \}$ such that:

$$\begin{array}{ll} \{f, g\} + \{g, f\} = 0 & \text{Antisymmetry} \\ \{f + g, h\} = \{f, h\} + \{g, h\} & \text{Linearity} \\ \{fg, h\} = f\{g, h\} + \{f, h\}g & \text{Product Law} \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 & \text{Jacobi Identity} \end{array} \tag{2.2.2.9}$$

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If the space is the phase space, then these Generalized Poisson Brackets, are reduced to the usual ones:

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \tag{2.2.2.10}$$

otherwise are subject to the previous algebra –only.

For a dynamical variable –say g, one can find –with the usage of:

$$\dot{g} = \frac{\partial g}{\partial q^i}\dot{q}^i + \frac{\partial g}{\partial p_i}\dot{p}_i \tag{2.2.2.11}$$

and of (2.2.2.8), as well as the generalized Poisson Bracket Algebra (2.2.2.9):

$$\dot{g} \approx \{g, H_T\} \tag{2.2.2.12}$$

The symbol $\approx$ is the *Weak Equality* symbol and stands for the following rule (deduced from the thorough analysis of the previous procedure):
A constraint, must not be used before all the Generalized Poisson Brackets, are calculated formally (i.e. only with the usage of the algebra (2.2.2.9) and the usual definition (2.2.2.10) –when the last is applicable). This rule, is encoded as:

$$\phi_m(q, p) \approx 0, \quad m \in [1, \dots, M] \tag{2.2.2.13}$$

In the previous procedure, the position of that rule, reflects the need to manipulate the u^m's, which may depend on t only –since they are unknown coefficients, the definition (2.2.2.10) can not be used.

If the dynamical variable g is any one of the constraints, then (2.2.2.5) declare the preservation of zero. Thus, consistency conditions, are deduced:

$$\{\phi_{m'}, H\} + u^m\{\phi_{m'}, \phi_m\} = 0 \tag{2.2.2.14}$$

There are three possibilities:

CC_1 Relations (2.2.2.14) lead to identities –maybe, with the help of (2.2.2.5).

CC_2 Relations (2.2.2.14) lead to equations independent of the u's. These turn to be constraints also. They called *Secondary*, but must be treated on the same footing as the primary ones.

CC_3 Relations (2.2.2.14) impose conditions on the u's.

The above procedure must be applied for all the secondary constraints. Again, the possible cases will be the previous three. The new constraints which may turn up are called secondary too. The procedure is applied for once more and so on. At the end, one will have a number of constraints (primary plus secondary) –say $\mathcal{J}$– and a number of conditions on the u's. A detailed analysis of the set of these conditions, shows that:

$$u^m = U^m(q, p) + \mathcal{V}^a(t)V_a^m(q, p) \tag{2.2.2.15}$$

where $V_a^m(q,p)$ are the a (in number) independent solutions of the homogeneous systems:

$$V_a^m(q,p)\{\phi_{m''},\phi_m\} = 0$$

The functions $\mathcal{V}^a(t)$ are related to the gauge freedom of the physical system.

Some terminology is needed at this point.

A dynamical variable R, is said to be *First Class*, if it has zero Poisson Bracket, with <u>all</u> the constraints:

$$\{R,\phi_n\} = 0, \quad n \in [1,\dots,\mathcal{J}] \tag{2.2.2.16}$$

where $\mathcal{J}$ is the total number of constraints –i.e. primary plus all the secondary ones. It is sufficient for these conditions, to hold weakly –since, by definition, the ϕ's are the only independent quantities that vanish weakly. Otherwise, the variable R, is said to be *Second Class*. If R, is First Class, then the quantity $\{R,\phi_n\}$ is strongly equal to some linear combination of the ϕ's. The following relative theorem (with a trivial proof) holds: *"The Poisson Bracket of two First Class quantities, is also First Class"*.

Using the result (2.2.2.15) the Hamiltonian (2.2.2.7), which is called *Total Hamiltonian*, is written:

$$H_T = H + U^m\phi_m + \mathcal{V}^a V_a^m \phi_m \equiv H' + \mathcal{V}^a\phi_a \tag{2.2.2.17}$$

with obvious associations. It can be proved that H' and ϕ_a, are first class quantities. With this splitting and the relation (2.2.2.12) for a dynamical variable g, it can be deduced that:

The First Class Primary Constraints ϕ_a, are the generating functions (i.e. the quantities $\{g,\phi_a\}$) of infinitesimal Contact Transformations; i.e. of transformations which lead to changes in the q's and the p's, but they do not affect the physical state of the system.

Successive application of two contact transformations generated by two given First Class Primary Constraints and taking into account the order, leads –for the sake of consistency– to a new generating function: $\{g,\{\phi_a,\phi_{a'}\}\}$. Thus one can see that First Class Secondary Constraints, which may turn up from $\{\phi_a,\phi_{a'}\}$, can also serve as generating functions of infinitesimal Contact Transformations. Possibly, another way to produce First Class Secondary Constraints, is the First Class quantity $\{H',\phi_a\}$. Since no one has found an example of a First Class Secondary Constraint, which affects the physical state when used as generating function, the conclusion is that all First Class quantities, are generating functions of infinitesimal Contact Transformations. Thus, the total Hamiltonian should be replaced by the *Extended Hamiltonian H_E*, defined as:

$$H_E = H_T + \mathcal{U}^{a''}\phi_{a''} \tag{2.2.2.18}$$

where the $\phi_{a''}$'s are those First Class Secondary Constraints, which are not already included in H_T. Finally, the equation of motion for a dynamical variable g (2.2.2.12) is altered:

$$\dot{g} \approx \{g,H_E\} \tag{2.2.2.19}$$

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2.3 Quantization of Constrained Systems

2.3.1 No Second Class Constraints are Present

The quantization of a classical physical system, whose Lagrangian, gives first class constraints only, is made in three steps:

S_1 The dynamical coordinates q's and momenta p's, are turned into Hermitian Operators $\widehat{q}$'s and $\widehat{p}$'s, satisfying the basic commutative algebra: $[\widehat{q}^i, \widehat{p}_j] = i\delta^i_j$.

S_2 A kind of a Schrödinger equation, is set up.

S_3 Any dynamical function, become Hermitian Operator –provided that the ordering problem is somehow solved.

Obviously, the constraints –being functions on the phase space– are subject to the S_3 rule. Dirac, proposed that when the constraints are turned into operators, they must annihilate the wave function Ψ:

$$\widehat{\phi}_i \Psi = 0, \quad \forall \quad i \tag{2.2.3.1}$$

Successive application of two such given conditions and taking into account the order, for sake of consistency, results in:

$$[\widehat{\phi}_i, \widehat{\phi}_j]\Psi = 0 \tag{2.2.3.2}$$

In order for operational conditions (2.2.3.2) not to give new ones on Ψ, one demands:

$$[\widehat{\phi}_i, \widehat{\phi}_j] = C^k_{ij}, \widehat{\phi}_k \tag{2.2.3.3}$$

If it is possible for such an algebra to be deduced, then no new operational conditions on Ψ are found and the system is consistent. If this is not the case, the new conditions must be taken into account and along with the initial ones, must give closed algebra, otherwise the procedure must be continued until a closed algebra is found. The discussion does not end here. Consistency between the operational conditions (2.2.3.1) and the Schrödinger equation, is pertinent as well. This lead to:

$$[\widehat{\phi}_i, \widehat{H}]\Psi = 0 \tag{2.2.3.4}$$

and consistency know, reads:

$$[\widehat{\phi}_i, \widehat{H}] = D^k_i, \widehat{\phi}_k \tag{2.2.3.5}$$

2.3.2 *Second Class Constraints are Present*

Suppose we have a classical physical system, whose Lagrangian, gives second class constraints. Any set of constraints, can be replaced by a corresponding set of independent linear combinations of them. It is thus, in principle, possible to make arrangement such that the final set of constraints, contains as much first class constraints as possible. Using the remaining –say S in number– second class constraints, the following matrix is defined:

$$\Delta_{ij} = [\chi_i, \chi_j], \quad (i,j) \in [1, \ldots, S] \tag{2.2.3.6}$$

where , χ's are the remaining (in classical form) second class constraints. A theorem can be proved :
"The determinant of this matrix does not vanish, not even weakly".

Since the determinant of Δ is non zero, there is the inverse of this matrix; say Δ^{-1}. Dirac, proposed a new kind of Poisson Bracket, the $\{\ ,\ \}_D$:

$$\{\ ,\ \}_D = \{\ ,\ \} - \sum_{i=1}^{S}\sum_{j=1}^{S} \{\ , \chi_i\} \Delta_{ij}^{-1} \{\chi_j,\ \} \tag{2.2.3.7}$$

These Brackets, are antisymmetric, linear in their arguments, obey the product law and the Jacobi identity. It holds that:

$$\{g, H_E\}_D \approx \{g, H_E\} \tag{2.2.3.8}$$

because terms like $\{\chi_i, H_E\}$, with H_E being first class, vanish weakly. Thus:

$$\dot{g} \approx \{g, H_E\}_D \tag{2.2.3.9}$$

But:

$$\{\xi, \chi_s\}_D = \ldots = 0 \tag{2.2.3.10}$$

if ξ is any of the q's or the p's. Thus, at the classical level, one may put the second class constraints equal to zero, before calculating the new Poisson Brackets. That means that:

M_1 The equations $\chi = 0$ may be considered as strong equations.

M_2 One, must ignore the corresponding degrees of freedom and

M_3 quantize the rest, according to the general rules, given in the previous section.

3 An Interdisciplinary Example: The Kantowski-Sachs Model

The purpose of the present section is to illustrate an application of the Dirac's method for constrained systems. The example chosen, is that of Kantowski-Sachs reduced Lagrangian –i.e. of a vacuum cosmological model; thus the interdisciplinary character of the section, emerges.

Consider, the following Lagrangian:

$$L = -\frac{a(t)\dot{b}^2(t) + 2b(t)\dot{a}(t)\dot{b}(t)}{2N(t)} + \frac{N(t)a(t)}{2} \tag{3.3.0.1}$$

where $a(t)$, $b(t)$ and $N(t)$, are the three degrees of freedom (the q's). It must be brought to the reader's notice that the Euler-Lagrange equations corresponding to this Lagrangian, are tantamount to the Einstein's Field Equations for the Kantowski-Sachs model (described in [8]), characterized by the line element:

$$ds^2 = -N^2(t)dt^2 + a^2(t)dr^2 + b^2(t)d\theta^2 + b^2(t)sin^2(\theta)d\phi^2 \tag{3.3.0.2}$$

The momenta are:

$$p_a = \frac{\partial L}{\partial \dot{a}(t)} = -\frac{b(t)\dot{b}(t)}{N(t)} \tag{3.3.0.3}$$

$$p_b = \frac{\partial L}{\partial \dot{b}(t)} = -\frac{\dot{a}(t)b(t) + a(t)\dot{b}(t)}{N(t)} \tag{3.3.0.4}$$

$$p_N = \frac{\partial L}{\partial \dot{N}(t)} = 0 \tag{3.3.0.5}$$

¿From the third of (3.3.0.3), one can see that there is one primary constraint:

$$p_N \approx 0 \tag{3.3.0.6}$$

The total Hamiltonian is:

$$H_T = H + u(a(t), b(t), N(t), t)p_N \tag{3.3.0.7}$$

where:

$$H = p_a\dot{a}(t) + p_b\dot{b}(t) - L = N(t)\Omega(t) \tag{3.3.0.8}$$

with:

$$\Omega(t) \equiv -\frac{a(t)}{2} - \frac{p_a p_b}{b(t)} + \frac{a(t)p_a^2}{2b^2(t)} \tag{3.3.0.9}$$

The consistency condition (2.2.2.14) applied to:

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A_1 the constraint $p_N \approx 0$, gives one secondary constraint:

$$\chi \equiv \{p_N, H\} = \{p_N, N(t)\Omega(t)\} = -\Omega(t) \approx 0 \tag{3.3.0.10}$$

A straightforward calculation, results in:

$$\{\chi, p_N\} = 0 \tag{3.3.0.11}$$

A_2 the previously deduced secondary constraint $\chi \approx 0$, gives –by virtue of (3.3.0.9), (3.3.0.10) and (3.3.0.11)– no further constraints, since it is identically satisfied (CC_1 case):

$$\{\chi, H\} + u(a(t), b(t), N(t), t)\{\chi, p_N\} = 0 \tag{3.3.0.12}$$

The Poisson Bracket (3.3.0.11) also declares that both p_N and χ, are first class quantities.

Finally, the equations of motion are:

$$\dot{a}(t) \approx \{a(t), H_T\} \tag{3.3.0.13}$$

$$\dot{p}_a \approx \{p_a, H_T\} \tag{3.3.0.14}$$

$$\dot{b}(t) \approx \{b(t), H_T\} \tag{3.3.0.15}$$

$$\dot{p}_b \approx \{p_b, H_T\} \tag{3.3.0.16}$$

$$\dot{N}(t) \approx \{N(t), H_T\} \tag{3.3.0.17}$$

$$\dot{p}_N \approx \{p_N, H_T\} \tag{3.3.0.18}$$

The first four equations constitute the usual set of the Euler-Lagrange equations for the $a(t)$ and $b(t)$, degrees of freedom. Equation (3.3.0.17), results in the gauge freedom related to $N(t)$ since –according to this equation– $\dot{N}(t) = u(a(t), b(t), N(t), t)$, i.e. an arbitrary function of time, while equation (3.3.0.18) is trivially satisfied, in view of (3.3.0.10).

4 Quantization of the most general Bianchi Type II, VI(Class A) & VII(Class A) Vacuum Cosmologies

We firstly treat of (see T. Christodoulakis, G. O. Papadopoulos, Phys. Lett. B **501** (2001) 264-8):

4.1 The Bianchi Type II Case

In [9], we had considered the quantization of an action corresponding to the most general Bianchi Type II cosmology, i.e. an action giving Einstein's Field Equations, derived from the line element:

$$ds^2 = (N^2(t) - N_a(t)N^a(t))dt^2 + 2N_a(t)\sigma_i^a(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma_i^\alpha(x)\sigma_j^\beta(x)dx^i dx^j \quad (4.4.1.1)$$

with:

$$
\begin{aligned}
\sigma^a(x) &= \sigma_i^\alpha(x)dx^i \\
\sigma^1(x) &= dx^2 - x^1 dx^3 \\
\sigma^2(x) &= dx^3 \\
\sigma^3(x) &= dx^1 \\
d\sigma^a(x) &= \tfrac{1}{2}C_{\beta\gamma}^a \sigma^\beta \wedge \sigma^\gamma \\
C_{23}^1 &= -C_{32}^1 = 1
\end{aligned}
\quad (4.4.1.2)
$$

see [10].

As is well known [11], the Hamiltonian is $H = \widetilde{N}(t)H_0 + N^a(t)H_a$ where:

$$H_0 = \frac{1}{2}L_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} + \gamma R \quad (4.4.1.3)$$

is the quadratic constraint with:

$$
\begin{aligned}
L_{\alpha\beta\mu\nu} &= \gamma_{\alpha\mu}\gamma_{\beta\nu} + \gamma_{\alpha\nu}\gamma_{\beta\mu} - \gamma_{\alpha\beta}\gamma_{\mu\nu} \\
R &= C_{\lambda\mu}^\beta C_{\theta\tau}^\alpha \gamma_{\alpha\beta}\gamma^{\theta\lambda}\gamma^{\tau\mu} + 2C_{\beta\delta}^\alpha C_{\nu\alpha}^\delta \gamma^{\beta\nu} + 4C_{\mu\nu}^\mu C_{\beta\lambda}^\beta \gamma^{\nu\lambda} = C_{\mu\kappa}^\alpha C_{\nu\lambda}^\beta \gamma_{\alpha\beta}\gamma^{\mu\nu}\gamma^{\kappa\lambda}
\end{aligned}
\quad (4.4.1.4)
$$

γ being the determinant of $\gamma_{\alpha\beta}$ (the last equality holding only for the Type II case), and:

$$H_a = C_{a\rho}^\mu \gamma_{\beta\mu}\pi^{\beta\rho} \quad (4.4.1.5)$$

are the linear constraints. Note that $\widetilde{N}$ appearing in the Hamiltonian, is to be identified with $N/\sqrt{\gamma}$.

The quantities H_0, H_a, are weakly vanishing [5], i.e. $H_0 \approx 0$, $H_a \approx 0$. For all class A Bianchi Types ($C_{\alpha\beta}^\alpha = 0$), it can be seen to obey the following first-class algebra:

$$
\begin{aligned}
\{H_0, H_0\} &= 0 \\
\{H_0, H_a\} &= 0 \\
\{H_a, H_\beta\} &= -\tfrac{1}{2}C_{\alpha\beta}^\gamma H_\gamma
\end{aligned}
\quad (4.4.1.6)
$$

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which ensures their preservation in time i.e. $\dot{H}_0 \approx 0$, $\dot{H}_a \approx 0$ and establishes the consistency of the action.

If we follow Dirac's general proposal [5] for quantizing this action, we have to turn H_0, H_a, into operators annihilating the wave function Ψ.

In the Schrödinger representation:

$$
\begin{aligned}
\gamma_{\alpha\beta} &\rightarrow \widehat{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} \\
\pi^{\alpha\beta} &\rightarrow \widehat{\pi}^{\alpha\beta} = -i\frac{\partial}{\partial\gamma_{\alpha\beta}}
\end{aligned}
\tag{4.4.1.7}
$$

satisfying the basic Canonical Commutation Relation (CCR) –corresponding to the classical ones:

$$
[\widehat{\gamma}_{\alpha\beta}, \widehat{\pi}^{\mu\nu}] = -i\delta^{\mu\nu}_{\alpha\beta} = \frac{-i}{2}(\delta^\mu_\alpha\delta^\nu_\beta + \delta^\mu_\beta\delta^\nu_\alpha)
\tag{4.4.1.8}
$$

The quantum version of the 2 independent linear constraints has been used to reduce, via the method of characteristics [12], the dimension of the initial configuration space from 6 ($\gamma_{\alpha\beta}$) to 4 (combinations of $\gamma_{\alpha\beta}$), i.e. $\Psi = \Psi(q, \gamma, \gamma^2_{12} - \gamma_{11}\gamma_{22}, \gamma_{12}\gamma_{13} - \gamma_{11}\gamma_{23})$, where $q = C^\alpha_{\mu\kappa}C^\beta_{\nu\lambda}\gamma_{\alpha\beta}\gamma^{\mu\nu}\gamma^{\kappa\lambda}$.

According to Kuhař's and Hajicek's [13] prescription, the ''kinetic'' part of H_0 is to be realized as the conformal Laplacian, corresponding to the reduced metric:

$$
L_{\alpha\beta\mu\nu}\frac{\partial x^i}{\partial\gamma_{\alpha\beta}}\frac{\partial x^j}{\partial\gamma_{\mu\nu}} = g^{ij}
\tag{4.4.1.9}
$$

where x^i, $i = 1, 2, 3, 4$, are the arguments of Ψ. The solutions had been presented in [9]. Note that the first-class algebra satisfied by H_0, H_a, ensures that indeed, all components of g^{ij} are functions of the x^i's. The signature of the g^{ij}, is $(+, +, -, -)$ signaling the existence of gauge degrees of freedom among the x^i's.

Indeed, one can prove [14] that the only gauge invariant quantity which, uniquely and irreducibly, characterizes a 3-dimensional geometry admitting the Type II symmetry group, is:

$$
q = C^\alpha_{\mu\kappa}C^\beta_{\nu\lambda}\gamma_{\alpha\beta}\gamma^{\mu\nu}\gamma^{\kappa\lambda}
\tag{4.4.1.10}
$$

An outline of the proof, is as follows:
Let two hexads $\gamma^{(1)}_{\alpha\beta}$ and $\gamma^{(2)}_{\alpha\beta}$ be given, such that their corresponding q's, are equal. Then [14] there exists an automorphism matrix Λ (i.e. satisfying $C^a_{\mu\nu}\Lambda^\kappa_a = C^\kappa_{\rho\sigma}\Lambda^\rho_\mu\Lambda^\sigma_\nu$) connecting them, i.e. $\gamma^{(1)}_{\alpha\beta} = \Lambda^\mu_\alpha\gamma^{(2)}_{\mu\nu}\Lambda^\nu_\beta$. But as it had been shown in the appendix of [15], this kind of changes on $\gamma_{\alpha\beta}$, can be seen to be induced by spatial diffeomorphisms. Thus, 3-dimensional Type II geometry, is uniquely characterized by some value of q.

Although for full pure gravity, Kuchař [16] has shown that there are not other first-class functions, homogeneous and linear in $\pi^{\alpha\beta}$, except H_a, imposing the extra symmetries (Type II), allows for such quantities to exist –as it will be shown. We are therefore, naturally led to seek the generators of these extra symmetries –which are

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expected to chop off x^2, x^3, x^4. Such quantities are, generally, called in the literature ''Conditional Symmetries''.

The automorphism group for Type II, is described by the following 6 generators –in matrix notation and collective form:

$$\lambda^a_{(I)\beta} = \begin{pmatrix} \kappa + \mu & x & y \\ 0 & \kappa & \rho \\ 0 & \sigma & \mu \end{pmatrix} \qquad (4.4.1.11)$$

with the property:

$$C^a_{\mu\nu}\lambda^\kappa_a = C^\kappa_{\mu\sigma}\lambda^\sigma_\nu + C^\kappa_{\sigma\nu}\lambda^\sigma_\mu \qquad (4.4.1.12)$$

¿From these matrices, we can construct the linear –in momenta– quantities:

$$A_{(I)} = \lambda^a_{(I)\beta}\gamma_{\alpha\rho}\pi^{\rho\beta} \qquad (4.4.1.13)$$

Two of these, are the H_a,'s since $C^a_{(\rho)\beta}$ correspond to the inner automorphism subgroup –designated by the x and y parameters, in $\lambda^a_{(I)\beta}$. The rest of them, are the generators of the outer automorphisms and are described by the matrices:

$$\varepsilon^a_{(I)\beta} = \begin{pmatrix} \kappa + \mu & 0 & 0 \\ 0 & \kappa & \rho \\ 0 & \sigma & \mu \end{pmatrix} \qquad (4.4.1.14)$$

The corresponding –linear in momenta– quantities, are:

$$E_{(I)} = \varepsilon^a_{(I)\beta}\gamma_{\alpha\rho}\pi^{\rho\beta} \qquad (4.4.1.15)$$

The algebra of these –seen as functions on the phase space, spanned by $\gamma_{\alpha\beta}$ and $\pi^{\mu\nu}$–, is:

$$\begin{aligned} \{E_I, E_J\} &= \widetilde{C}^K_{IJ}E_K \\ \{E_I, H_a\} &= -\tfrac{1}{2}\lambda^\beta_a H_\beta \\ \{E_I, H_0\} &= -2(\kappa + \mu)\gamma R \end{aligned} \qquad (4.4.1.16)$$

¿From the last of (4.4.1.16), we conclude that the subgroup of E_I's with the property $\kappa + \mu = 0$, i.e. the traceless generators, are first-class quantities; their time derivative vanishes. So let:

$$\widetilde{E}_I = \{E_I : \ \kappa + \mu = 0\} \qquad (4.4.1.17)$$

Then, the previous statement translates into the form:

$$\dot{\widetilde{E}}_I = 0 \Rightarrow \widetilde{E}_I = c_I \qquad (4.4.1.18)$$

the c_I's being arbitrary constants.

Now, these are –in principle– integrals of motion. Since, as we have earlier seen, $\widetilde{E}_I$'s along with H_a's, generate automorphisms, it is natural to promote the integrals of motion (4.4.1.18), to symmetries –by setting the c_I's zero. The action of the quantum version of these $\widetilde{E}_I$'s on Ψ, is taken to be [13]:

$$\left.\begin{array}{l} \widehat{\widetilde{E}}_I \Psi = \varepsilon^a_{(I)\beta} \gamma_{\alpha\rho} \frac{\partial \Psi}{\gamma_{\beta\rho}} = 0 \\[2mm] \varepsilon^a_{(I)a} = 0 \end{array}\right\} \Rightarrow \Psi = \Psi(q, \gamma) \tag{4.4.1.19}$$

The Wheeler-DeWitt equation now, reads:

$$5q^2 \frac{\partial^2 \Psi}{\partial q^2} - 3\gamma^2 \frac{\partial^2 \Psi}{\partial \gamma^2} + 2q\gamma \frac{\partial^2 \Psi}{\partial \gamma \partial q} + 5q \frac{\partial \Psi}{\partial q} - 3\gamma \frac{\partial \Psi}{\partial \gamma} - 2q\gamma \Psi = 0 \tag{4.4.1.20}$$

Note that:

$$\nabla^2_c = \nabla^2 + \frac{(d-2)}{4(d-1)} R = \nabla^2 \tag{4.4.1.21}$$

since we have a 2-dimensional, flat space, with contravariant metric:

$$g^{ij} = \begin{pmatrix} 5q^2 & q\gamma \\ q\gamma & -3\gamma^2 \end{pmatrix} \tag{4.4.1.22}$$

which is Lorentzian. This equation, can be easily solved by separation of variables; transforming to new coordinates $u = q\gamma^3$ and $v = q\gamma$, we get the 2 independent equations:

$$\begin{array}{l} 16u^2 A''(u) + 16u A'(u) - cA(u) = 0 \\[2mm] B''(v) + \frac{1}{v} B'(v) - (\frac{1}{2v} + \frac{c}{4v^2}) B(v) = 0 \end{array} \tag{4.4.1.23}$$

where c, is the separation constant. Equation (4.4.1.20), is of hyperbolic type and the resulting wave function will still not be square integrable. Besides that, the tracefull generators of the outer automorphisms, are left inactive –due to the non vanishing CCR with H_0.

These two facts, lead us to deduce that there must still exist a gauge symmetry, corresponding to some –would be, linear in momenta– first-class quantity. Our starting point in the pursuit of this, is the third of (4.4.1.16). It is clear that we need another quantity –also linear in momenta– with an analogous property; the trace of $\pi^{\mu\nu}$, is such an object. We thus define the following quantity:

$$T = E_I - (\kappa + \mu)\gamma_{\alpha\beta}\pi^{\alpha\beta} \tag{4.4.1.24}$$

in the phase space –spanned by $\gamma_{\alpha\beta}$ and $\pi^{\mu\nu}$. It holds that:

$$\begin{array}{l} \{T, H_0\} = 0 \\ \{T, H_a\} = 0 \\ \{T, E_I\} = 0 \end{array} \tag{4.4.1.25}$$

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because of:

$$\{E_I, \gamma\} = -2(\kappa + \mu)\gamma$$
$$\{E_I, q\} = 0$$
$$\gamma_{\alpha\beta}\{\pi^{\alpha\beta}, q\} = q$$
$$\gamma_{\alpha\beta}\{\pi^{\alpha\beta}, \gamma\} = -3\gamma$$

$$(4.4.1.26)$$

Again –as for $\widetilde{E}_I$'s–, we see that since T, is first-class, we have that:

$$\dot{T} = 0 \Rightarrow T = const = c_T \tag{4.4.1.27}$$

another integral of motion. We therefore see, that T has all the necessary properties to be used in lieu of the tracefull generator, as a symmetry requirement on Ψ. In order to do that, we ought to set c_T zero –exactly as we did with the c_I's, corresponding to $\widetilde{E}_I$'s. The quantum version of T, is taken to be:

$$\widehat{T} = \lambda_\beta^\alpha \gamma_{\alpha\rho} \frac{\partial}{\partial \gamma_{\beta\rho}} - (\kappa + \mu)\gamma_{\alpha\beta}\frac{\partial}{\partial \gamma_{\alpha\beta}} \tag{4.4.1.28}$$

Following, Dirac's theory, we require:

$$\widehat{T}\Psi = \lambda_\beta^\alpha \gamma_{\alpha\rho} \frac{\partial \Psi}{\partial \gamma_{\beta\rho}} - (\kappa + \mu)\gamma_{\alpha\beta}\frac{\partial \Psi}{\partial \gamma_{\alpha\beta}} = (\kappa + \mu)(q\frac{\partial \Psi}{\partial q} - \gamma\frac{\partial \Psi}{\partial \gamma}) = 0 \tag{4.4.1.29}$$

Equation (4.4.1.29), implies that $\Psi(q, \gamma) = \Psi(q\gamma)$ and thus equation (4.4.1.20), finally, reduces to:

$$4w^2\Psi''(w) + 4w\Psi'(w) - 2w\Psi = 0 \tag{4.4.1.30}$$

where, for simplicity, $w \doteq q\gamma$. The solution to this equation, is:

$$\Psi = c_1 I_0(\sqrt{2q\gamma}) + c_2 K_0(\sqrt{2q\gamma}) \tag{4.4.1.31}$$

where I_0 is the modified Bessel function, of the first kind, and K_0 is the modified Bessel function, of the second kind, both with zero argument.

At first sight, it seems that although we have apparently exhausted the symmetries of the system, we have not yet been able to obtain a wave function on the space of the 3-geometries, since Ψ depends on $q\gamma$ and not on q only. On the other hand, the fact that we have achieved a reduction to one degree of freedom, must somehow imply that the wave function found must be a function of the geometry. This puzzle finds its resolution as follows. Consider the quantity:

$$\Omega = -2\gamma_{\rho\sigma}\pi^{\rho\sigma} + \frac{2C_{\mu\kappa}^a C_{\nu\lambda}^\beta \gamma^{\kappa\lambda}\gamma^{\mu\nu}\gamma_{\alpha\rho}\gamma_{\beta\sigma} - 4C_{\mu\rho}^\alpha C_{\nu\sigma}^\beta \gamma_{\alpha\beta}\gamma^{\mu\nu}}{q}\pi^{\rho\sigma} \tag{4.4.1.32}$$

This can also be seen to be first-class, i.e.

$$\dot{\Omega} = 0 \Rightarrow \Omega = const = c_\Omega \tag{4.4.1.33}$$

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Moreover, it is a linear combination of T, $\widetilde{E}_I$'s, and H_a's, and thus $c_\Omega = 0$. Now it can be verified that Ω, is nothing but:

$$\frac{1}{N(t)}\left(\frac{\dot{\gamma}}{\gamma} + \frac{1}{3}\frac{\dot{q}}{q}\right) \tag{4.4.1.34}$$

So:

$$\gamma q^{1/3} = \vartheta = constant \tag{4.4.1.35}$$

Without any loss of generality, and since ϑ is not an essential constant of the classical system (see [17] and reference [18] therein), we set $\vartheta = 1$. Therefore:

$$\Psi = c_1 I_0(\sqrt{2}q^{1/3}) + c_2 K_0(\sqrt{2}q^{1/3}) \tag{4.4.1.36}$$

where I_0 is the modified Bessel function, of the first kind, and K_0 is the modified Bessel function, of the second kind, both with zero argument.

As for the measure, it is commonly accepted that, there is not a unique solution. A natural choice, is to adopt the measure that makes the operator in (4.4.1.30), hermitian –that is:

$$\mu(q) \propto q^{-1} \tag{4.4.1.37}$$

It is easy to find combinations of c_1 and c_2 so that the probability $\mu(q)|\Psi|^2$, be defined.

Note that putting the constant associated with Ω, equal to zero, amounts in restricting to a subset of the classical solutions, since c_Ω, is one of the two essential constants of Taub's solution. One could keep that constant, at the expense of arriving at a wave function with explicit time dependence, since then:

$$\gamma = q^{-1/3} Exp\left[\int c_\Omega N(t)dt\right] \tag{4.4.1.38}$$

We however, consider more appropriate to set that constant zero, thus arriving at a Ψ depending on q only, and decree its applicability to the entire space of the classical solutions. Anyway this is not such a blunder, since Ψ is to give weight to all states, –being classical ones, or not.

The above treatise, remains almost unchanged for –(T. Chistodoulakis, G. O. Papadopoulos, to appear in Phys. Lett. B):

4.2 *The Class A Bianchi Type VI & VII Case*

The only difference between the various Bianchi Types, is that of the values of the structure constants. Of course, it must be emphasized that in the present work we study differential equations (i.e. the Einstein's Field Equations), which are defined locally –we are not concerned with global properties (i.e. topological aspects).

It is well known that in 3 dimensions, the tensor $C^\alpha_{\beta\gamma}$ admits a unique decomposition in terms of a contravariant symmetric tensor density of weight -1, $m^{\alpha\beta}$, and a covariant vector $\nu_\alpha = \frac{1}{2}C^\rho_{\alpha\rho}$ as follows:

$$C^\alpha_{\beta\gamma} = m^{\alpha\delta}\varepsilon_{\delta\beta\gamma} + \nu_\beta\delta^\alpha_\gamma - \nu_\gamma\delta^\alpha_\beta \tag{4.4.2.1}$$

For the Class A ($\nu_\alpha = 0$) Bianchi Type VI ($\varepsilon = 1$) and VII ($\varepsilon = -1$), this matrix is:

$$m^{\alpha\beta} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.4.2.2}$$

resulting in the following non vanishing structure constants ([10]):

$$C^1_{23} = \varepsilon \qquad C^2_{13} = 1 \tag{4.4.2.3}$$

In general, via the method of characteristics [12], the quantum version of the three independent linear constraints can be used to reduce the dimension of the initial configuration space from 6 ($\gamma_{\alpha\beta}$) to 3 (combinations of $\gamma_{\alpha\beta}$), i.e. $\Psi = \Psi(q^1, q^2, q^3)$ [13], where:

$$\begin{aligned} q^1 &= C^\alpha_{\mu\kappa}C^\beta_{\nu\lambda}\gamma^{\mu\nu}\gamma^{\kappa\lambda}\gamma_{\alpha\beta} \\ q^2 &= C^\alpha_{\beta\kappa}C^\beta_{\alpha\lambda}\gamma^{\kappa\lambda} \\ q^3 &= \gamma \end{aligned} \tag{4.4.2.4}$$

In analogy with the preceding case (where an outline of the proof is exhibited), the only G.C.T. (gauge) invariant quantities, which uniquely and irreducibly, characterize a 3-dimensional geometry admitting the Class A Type VI and VII symmetry groups, are the quantities q^1 and q^2.

In terms of the three q 's, one can define the following –according to [13]– induced ''physical'' metric, given by the relation:

$$g^{ij} = L_{\alpha\beta\mu\nu}\frac{\partial q^i}{\partial\gamma_{\alpha\beta}}\frac{\partial q^j}{\partial\gamma_{\mu\nu}} = \begin{pmatrix} 5q^1q^1 - 16q^2q^2 & q^1q^2 & q^1q^3 \\ q^1q^2 & q^2q^2 & q^2q^3 \\ q^1q^3 & q^2q^3 & -3q^3q^3 \end{pmatrix} \tag{4.4.2.5}$$

Note that the first-class algebra satisfied by H_0, H_α, ensures that indeed, all components of g^{ij} are functions of the q^i.

The automorphism group for the Class A Type VI and VII, is described by the following 4 generators –in matrix notation and collective form:

$$\lambda^\alpha_{(I)\beta} = \begin{pmatrix} a & \varepsilon\, c & b \\ c & a & d \\ 0 & 0 & 0 \end{pmatrix} \tag{4.4.2.6}$$

¿From these matrices, we can construct the linear –in momenta– quantities:

$$A_{(I)} = \lambda^\alpha_{(I)\beta}\gamma_{\alpha\rho}\pi^{\rho\beta} \tag{4.4.2.7}$$

Three of these are the H_α's, since $C^\alpha_{(\rho)\beta}$ correspond to the inner automorphism subgroup –designated by the c, b and d parameters in $\lambda^\alpha_{(I)\beta}$. The remaining is the generator of the outer automorphisms and is given by the essentially unique matrix

$$\epsilon^\alpha_\beta = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.4.2.8}$$

The corresponding –linear in momenta– quantity is:

$$E = \epsilon^\alpha_\beta \gamma_{\alpha\rho}\pi^{\rho\beta} \tag{4.4.2.9}$$

It is straightforward to calculate the Poisson Brackets of E with H_0, H_α:

$$\begin{aligned} \{E, H_\alpha\} &= -\tfrac{1}{2}\lambda^\beta_\alpha H_\beta \\ \{E, H_0\} &= -2a\gamma R = -2a\gamma(q^1 + 2q^2) \end{aligned} \tag{4.4.2.10}$$

At this point, it is crucial to observe that we can construct a classical integral of motion, i.e. an extra gauge symmetry of the corresponding classical action: notice that the trace of the canonical momenta, $\gamma_{\mu\nu}\pi^{\mu\nu}$, has vanishing PB with H_α, E, and a –similar to E– PB with H_0 equal to $2\gamma R$; thus, if we define:

$$T = E - a\gamma_{\mu\nu}\pi^{\mu\nu} \tag{4.4.2.11}$$

we can easily derive the following PB of T with H_0, H_α:

$$\begin{aligned} \{T, H_0\} &= 0 \\ \{T, H_1\} &= -\tfrac{a}{2}H_1 \approx 0 \\ \{T, H_2\} &= -\tfrac{a}{2}H_2 \approx 0 \\ \{T, H_3\} &= 0 \end{aligned} \tag{4.4.2.12}$$

The quantity T, is thus revealed to be first-class, and therefore an integral of motion (since the Hamiltonian, is a linear combination of the constraints):

$$\dot{T} = \{T, H\} \approx 0 \Rightarrow T = \text{const} = C_T \tag{4.4.2.13}$$

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The quantum version of T, is taken to be [13]:

$$\widehat{T} = \epsilon^{\alpha}_{\beta} \gamma_{\alpha\rho} \frac{\partial}{\partial \gamma_{\beta\rho}} - a\gamma_{\alpha\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \qquad (4.4.2.14)$$

(without any loss of generality –see (4.4.2.8)– we can safely suppress a, whenever convenient, by setting it to 1).

Following the spirit of Dirac, we require:

$$\widehat{T}\Psi = \epsilon^{\alpha}_{\beta} \gamma_{\alpha\rho} \frac{\partial \Psi}{\partial \gamma_{\beta\rho}} - a\gamma_{\alpha\beta} \frac{\partial \Psi}{\partial \gamma_{\alpha\beta}} = (q^1 \frac{\partial \Psi}{\partial q^1} + q^2 \frac{\partial \Psi}{\partial q^2} - \gamma \frac{\partial \Psi}{\partial \gamma}) = C_T \Psi \qquad (4.4.2.15)$$

The general solution [12] to the above equation has the form:

$$\Psi = \gamma^{-C_T} \Theta(\frac{q^1}{q^2}, \gamma q^2) \qquad (4.4.2.16)$$

Θ being an arbitrary function in its arguments.

Now, the number of our dynamical variables, is reduced from 3 $(q^i, i = 1, 2, 3)$ to 2, namely the combinations $w^1 = q^1/q^2$ and $w^2 = q^2 q^3$. So, we have a further reduction of the 3-dimensional configuration space spanned by the 3 q 's. Again, in terms of the w 's, the finally reduced ''physical'' –although singular– metric, is given by the following relation:

$$s^{kl} = g^{ij} \frac{\partial w^k}{\partial q^i} \frac{\partial w^l}{\partial q^j} = \begin{pmatrix} -16 + 4(w^1)^2 & 0 \\ 0 & 0 \end{pmatrix} \qquad (4.4.2.17)$$

The singular character of this metric is not unexpected; its origin lies in the fact that $L^{\alpha\beta\gamma\delta}T_{\alpha\beta}T_{\gamma\delta} = 0$, where $L^{\alpha\beta\gamma\delta} = \frac{1}{4}(\gamma^{\alpha\gamma}\gamma^{\beta\delta} + \gamma^{\alpha\delta}\gamma^{\beta\gamma} - 2\gamma^{\alpha\beta}\gamma^{\gamma\delta})$ is the covariant supermetric (inverse to $L_{\alpha\beta\gamma\delta}$) and $T_{\alpha\beta}$ are the components of T, seen as vector field in the initial superspace spanned by $\gamma_{\alpha\beta}$. Indeed, it is known that reducing to null surfaces entails all sort of peculiarities.

So far, the degrees of freedom are two (w^1, w^2). The vanishing of the s^{22} component indicates that w^2 is not dynamical at the quantum level. This fact has its analogue at the classical level; indeed, consider the quantity:

$$\Omega = \frac{(\gamma q^2)^{\cdot}}{\gamma q^2} = L_{\alpha\beta\mu\nu}\gamma^{\mu\nu}\pi^{\alpha\beta} - L_{\kappa\lambda\rho\sigma}\pi^{\rho\sigma} \frac{C^{\alpha}_{\beta\mu}C^{\beta}_{\alpha\nu}\gamma^{\mu\kappa}\gamma^{\nu\lambda}}{q^2} \qquad (4.4.2.18)$$

In (4.4.2.18), the transition from velocity phase space to momentum phase space has been made using the Hamiltonian (4.4.1.3) –with the proper R. It is straightforward to verify that:

$$\Omega = \frac{2}{a}T - \frac{4\varepsilon\gamma^{23}}{q^2}H_1 - \frac{4\varepsilon\gamma^{13}}{q^2}H_2 \qquad (4.4.2.19)$$

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Taking into account the weak vanishing of the linear constraints, it is deduced that;

$$\Omega = \frac{2}{a}T = \frac{2C_T}{a} = 2C_T \qquad (4.4.2.20)$$

if we set $a = 1$.

Another way to show that Ω is constant, is the following:

$$\{\Omega, H\} = \{\frac{\{\gamma q^2, H\}}{\gamma q^2}, H\} = \frac{4\gamma}{(\gamma q^2)^2}(4\gamma_{11}H_1^2 + 4\gamma_{22}H_2^2 - 8\varepsilon\gamma_{12}H_1H_2) \qquad (4.4.2.21)$$

Again, the weak vanishing of the linear constraints, ensures that Ω is constant.

Using (4.4.2.18), (4.4.2.20) and the action, we have:

$$\gamma q^2 = C_1 exp\{2C_T \int \widetilde{N}(t)dt\} \qquad (4.4.2.22)$$

Now, returning to the quantum domain, we observe that out of the three arguments of the wave function given in (4.4.2.16), only q^1/q^2 is G.C.T. invariant –in the sense previously explained. This suggests that we must somehow eliminate γ, γq^2. To this end, we adopt the value zero for the classical constant C_T. This amounts to restricting to a 2-parameter subspace of the classical space of solutions, spanned by the 3 essential constants [17]. This means that we base our quantum theory on this subspace and decree the wave function, to be applicable to all configurations (classical or not). The benefit of such an action, is twofold: γ^{-C_T} drops out, while at the same time $w^2 \equiv \gamma q^2$ is set equal to the constant C_1 –see (4.4.2.22). These facts, along with the obligation that no derivatives with respect to w^2, are to enter the Wheeler-DeWitt equation –see (4.4.2.17)–, allow us to arrive at the following form for the wave function (4.4.2.16):

$$\Psi = \Theta(\frac{q^1}{q^2}, C_1) \qquad (4.4.2.23)$$

and of course:

$$w^2 = q^2q^3 = \text{const} = C_1 \qquad (4.4.2.24)$$

Now, the final reduction of the configuration space is achieved. Our dynamic variable is the ratio q^1/q^2, which is a combination of the only curvature invariants existing in Class A Bianchi Type VI and VII, and emerges as the only true quantum degree of freedom.

Consequently, following the spirit of [13], we have to construct the quantum analogue of H_0 as the conformal Laplacian, based on the non-singular part of the ''physical'' metric (4.4.2.17), i.e.:

$$\widehat{H}_0 = -\frac{1}{2}\nabla_c^2 + w^2(2 + w^1) \qquad (4.4.2.25)$$

where:

$$\nabla_c^2 = \nabla^2 = \frac{1}{\sqrt{s_{11}}} \partial_{w^1}\{\sqrt{s_{11}} \, s^{11} \, \partial_{w^1}\} \qquad (4.4.2.26)$$

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is the 1-dimensional Laplacian based on s_{11} ($s^{11}s_{11} = 1$). Note that in 1-dimension the conformal group is totally contained in the G.C.T. group, in the sense that any conformal transformation of the metric can not produce any change in the –trivial– geometry and is thus reachable by some G.C.T. Therefore, no extra term in needed in (4.4.2.26), as it can also formally be seen by taking the limit $d = 1$, $R = 0$ in the general definition:

$$\nabla_c^2 \equiv \nabla^2 + \frac{(d-2)}{4(d-1)}R = \nabla^2 \qquad (4.4.2.27)$$

Thus, the Wheeler-DeWitt equation, reads:

$$\widehat{H}_0\Psi = \sqrt{(w^2 - 4)}\, \partial_w\{\sqrt{(w^2 - 4)}\, \partial_w\}\Psi - \frac{C_1(2+w)}{2}\Psi = 0 \qquad (4.4.2.28)$$

where, for simplicity, we set $w^1 \equiv w$ and $w^2 = C_1$.
Using the transformation $w = 2cosh(z)$, the previous equation takes the form:

$$\frac{\partial^2\Psi}{\partial z^2} - (C_1 + cosh(z))\Psi = 0 \qquad (4.4.2.29)$$

The solutions to this family of equations are the Mathieu Modified Functions –see [18] and references therein, for an extended treatise, in various cases.

As for the measure, it is commonly accepted that there is not a unique solution. A natural choice, is to adopt the measure that makes the operator in (4.4.2.28) hermitian, that is $\mu(w) = \frac{1}{\sqrt{w^2-4}}$, or in the variable z, $\mu(z) = 1$. However, the solutions to (4.4.2.29) can be seen to violently diverge for various values of $z \in [0, \infty)$, which is the classically allowed region. If we wish to avoid this difficulty, we can abandon hermiticity, especially in view of the fact that we are interested in the zero eigenvalues of the operator, and thus does not make any harm to lose realness of the eigenvalues. If we adopt this attitude, we can find suitable measures, e.g. $\mu(z) = e^{-z^2}$. The probability density $\rho(z) = \mu(z)|\Psi(z)|^2$ is now finite, enabling one to assign a number between 0 and 1 to each Class A Type VI and VII geometry.
Another feature of the reduction to (4.4.2.29) is that the final dynamical argument of Ψ is the ratio q^1/q^2, which is of degree zero in the scale factors, as seen from equation (4.4.2.4). This consists a kind of build-in regularization with respect to the volume of the 3-space. Moreover, the solutions to this equation exhibit an increasingly oscillatory behavior, as C_1 increases. This is most welcomed and expected in view of C_1 being some kind of measure of the 3-volume, since it contains γ –see (4.4.2.24).

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